Exam Calculus 2

9 April 2018, 9:00-12:00

university of groningen

The exam consists of 6 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. **[6+6+3=15 Points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) For which unit vector $\boldsymbol{u} = v\mathbf{i}+w\mathbf{j}$ with $v^2+w^2 = 1$, does the directional derivative $D_{\boldsymbol{u}}f(0,0)$ exist?

   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. **[7+8=15 Points.]** Let $u : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto u(x,y,z)$ be a C^2 function. By defining spherical coordinates according to $(x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, the function $u(x,y,z)$ can be considered as a function $f(\rho,\theta,\phi)$.

   (a) Express $\frac{\partial f}{\partial \theta}$ in terms of partial derivatives with respect to x, y and z of the function u.

   (b) Conversely the function $f(\rho,\theta,\phi)$ can be considered as a function $u(x,y,z)$. Suppose that the function f depends only on ρ (i.e. f is independent of θ and ϕ). Show that in this case

$$\frac{\partial^2}{\partial x^2}u(x,y,z) + \frac{\partial^2}{\partial y^2}u(x,y,z) + \frac{\partial^2}{\partial z^2}u(x,y,z) = \frac{2}{\rho}f'(\rho) + f''(\rho).$$

3. **[4+4+7=15 Points.]** Consider the helix parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k},$$

   where a and b are positive constants.

   (a) Determine the length of the helix and its parametrization by arclength s.

   (b) At each point on the helix, determine the unit tangent vector $\mathbf{T}$ and the curvature of the helix κ.

   (c) Let $\mathbf{N}$ be the unit vector with direction $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{T}$ and let $\mathbf{B}$ be the unit vector defined as $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Compute $\mathbf{B}$ and show that $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{B} = -\tau\mathbf{N}$ for some $\tau \in \mathbb{R}$. Determine τ.

— **please turn over** —

4. **[3+6+6=15 Points]** Let S be the unit sphere in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$.

   (a) Compute the tangent plane of S at the point $(x_0, y_0, z_0) = (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$.

   (b) Use the Implicit Function Theorem to show that near the point $(x_0, y_0, z_0) = (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, the sphere S can be considered to be the graph of a function f of x and y. Compute the partial derivatives of f with respect to x and y and show that the tangent plane found in (a) coincides with the graph of the linearization of f at $(x_0, y_0) = (1/\sqrt{3}, 1/\sqrt{3})$.

   (c) Use the method of Lagrange multipliers to determine the points on S where $f(x, y, z) = xy^2z^3$ has maxima and minima, respectively.

5. **[5+5+5=15 Points]** For constants $a, b \in \mathbb{R}$, define the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\mathbf{F}(x, y, z) = ax \sin(\pi y)\, \mathbf{i} + (x^2 \cos(\pi y) + bye^{-z})\, \mathbf{j} + y^2 e^{-z}\, \mathbf{k}.$$

   (a) Show that $\mathbf{F}$ to be conservative requires $a = 2/\pi$ and $b = -2$.

   (b) Determine a scalar potential for $\mathbf{F}$ for the values of a and b given in part (a).

   (c) For the values of a and b given in part (a), compute the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the curve parametrized by

$$\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin^2 t\, \mathbf{j} + \sin(2t)\, \mathbf{k}$$

   with $t \in [0, \pi/2]$.

6. **[8+7=15 Points]** For $r > 0$, let S_r denote the sphere of radius r with center at the origin, oriented with outward normal. Suppose $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is of class C^1 and is such that

$$\oint\!\!\!\!\oint_{S_r} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = ar + b \tag{1}$$

for fixed constants a and b.

   (a) Compute

$$\iiint_D \nabla \cdot \mathbf{F}\, \mathrm{d}V,$$

   where $D = \{(x, y, z) \in \mathbb{R}^3 \mid 25 \le x^2 + y^2 + z^2 \le 49\}$.

   (b) Suppose that $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ of class C^2 and Equation (1) holds for any $r > 0$. What conditions does this place on the constants a and b?

1. a) $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \quad \frac{x^2-y^2}{x^2+y^2} = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \cos^2\theta - \sin^2\theta$$

which for example equals 1 for $\theta = 0$

and -1 for $\theta = \frac{\pi}{2}$.

So f has no limit at $(x,y) = (0,0)$ and

hence cannot be continuous at $(x,y) = (0,0)$.

b) $$\frac{f(tv,tw) - f(0,0)}{t} = \frac{\frac{t^2v^2 - t^2w^2}{t^2v^2+t^2w^2} - 0}{t}$$

$= \frac{1}{t}(v^2 - w^2)$. For this to have a limit for $t \to 0$,

$v^2 = w^2$, i.e. $\quad v = \pm\frac{1}{\sqrt{2}}$, $w = \pm\frac{1}{\sqrt{2}}$ (independently)

c) f is not differentiable at $(0,0)$ as f is not continuous at $(0,0)$.

2\. a) $$\frac{\partial f}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial u}{\partial z}$$

$$= -\rho \sin\phi \sin\theta \frac{\partial u}{\partial x} + \rho \sin\phi \cos\theta \frac{\partial u}{\partial z}$$

$$= -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}$$

b) $$\frac{\partial u}{\partial x} = \frac{\partial \rho}{\partial x}\frac{df}{d\rho} = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{df}{d\rho}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{df}{d\rho}\right) = \frac{x^2}{x^2+y^2+z^2}\frac{d^2 f}{d\rho^2}$$

$$+ \frac{df}{d\rho}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{x^2}{x^2+y^2+z^2}\frac{d^2f}{d\rho^2} + \frac{df}{d\rho}\frac{\sqrt{x^2+y^2+z^2} - \frac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$= \frac{x^2}{r^2}\frac{d^2f}{d\rho^2} + \frac{df}{d\rho}\frac{r - \frac{x^2}{r}}{r^2}$$

similarly for y and z.

Summing the terms for x, y and z gives

$$\frac{d^2f}{d\rho^2} + \left(\frac{3}{r} - \frac{r^2}{r^3}\right)\frac{df}{d\rho} = \frac{2}{\rho}f'(\rho) + f''(\rho)$$

3. (a) $r'(t) = -a\sin t\, i + a\cos t\, j + bk \Rightarrow$ arclength

$$s(t) = \int_0^t \|r'(\tau)\| d\tau = \int_0^t (a^2+b^2)^{1/2} d\tau = t(a^2+b^2)^{1/2}$$

$s(2\pi) = 2\pi(a^2+b^2)^{1/2}$ = length of the helix

parametrization by arclength:

$$\tilde{r}(s) = a\cos\frac{s}{\sqrt{a^2+b^2}}\, i + a\sin\frac{s}{\sqrt{a^2+b^2}}\, j + \frac{b}{\sqrt{a^2+b^2}}\, s k$$

$s \in [0, 2\pi(a^2+b^2)^{1/2}]$

b) unit tangent vector:

$$T = \frac{d\tilde{r}}{ds} = -\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}\, i + \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}\, j + \frac{b}{\sqrt{a^2+b^2}}\, k$$

$$\frac{dT}{ds} = -\frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}\, i - \frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}\, j + 0\cdot k$$

curvature:

$$K = \left\|\frac{dT}{ds}\right\| = \left(\frac{a^2}{(a^2+b^2)^2}\right)^{1/2} = \frac{a}{a^2+b^2}$$

c) $$N = \frac{a^2+b^2}{a}\left(-\frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}\, i - \frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}\, j\right)$$

$$= -\cos\frac{s}{\sqrt{a^2+b^2}}\, i - \sin\frac{s}{\sqrt{a^2+b^2}}\, j$$

$$B = T \times N = \begin{vmatrix} i & j & k \\ -\frac{a}{\sqrt{a^2+b^2}}\sin & \frac{a}{\sqrt{a^2+b^2}}\cos & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos & -\sin & 0 \end{vmatrix}$$

$$= +\frac{b}{\sqrt{a^2+b^2}}\sin i - \frac{b}{\sqrt{a^2+b^2}}\cos j + \frac{a}{\sqrt{a^2+b^2}}\, k$$

$$\frac{dB}{ds} = +\frac{b}{a^2+b^2}\cos i + \frac{b}{a^2+b^2}\sin j \overset{!}{=} -\tau N \Rightarrow \tau = \frac{b}{a^2+b^2}$$

4. a) $F(x,y,z) = x^2+y^2+z^2 \Rightarrow$ unit sphere: $F(x,y,z) = 1$

$\nabla F(x,y,z) = 2x\,i + 2y\,j + 2z\,k$ normal to S at any point $(x,y,z) \in S$.

tangent plane:

$$\nabla F(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$\Leftrightarrow \quad 2\tfrac{1}{\sqrt{3}}\left(x-\tfrac{1}{\sqrt{3}}\right) + 2\tfrac{1}{\sqrt{3}}\left(y-\tfrac{1}{\sqrt{3}}\right) - 2\tfrac{1}{\sqrt{3}}\left(z+\tfrac{1}{\sqrt{3}}\right) = 0$$

$$\Leftrightarrow \quad x - \tfrac{1}{\sqrt{3}} + y - \tfrac{1}{\sqrt{3}} + z + \tfrac{1}{\sqrt{3}} = 0$$

$$\Leftrightarrow \quad x+y+z = \tfrac{1}{\sqrt{3}}$$

b) For F in part (a) we have $\frac{\partial F}{\partial z}(x_0,y_0,z_0) = -\frac{2}{\sqrt{3}} \neq 0$.

By IFT, S is locally the graph of a function $f:(x,y) \mapsto z$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)} = -\left.\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right|_{(x,y,z)=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)} = -\frac{2\frac{1}{\sqrt{3}}}{-\frac{2}{\sqrt{3}}} = 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)} = -\left.\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right|_{(x,y,z)=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)} = 1$$

$\Rightarrow$ Linearization of f at $(x_0,y_0) = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ is

$$L(x,y) = f\left(\tfrac{1}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\right) + f_x\left(\tfrac{1}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\right)\left(x-\tfrac{1}{\sqrt{3}}\right) + f_y\left(\tfrac{1}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\right)\left(y-\tfrac{1}{\sqrt{3}}\right)$$

$$= -\tfrac{1}{\sqrt{3}} + \left(x-\tfrac{1}{\sqrt{3}}\right) + \left(y-\tfrac{1}{\sqrt{3}}\right)$$ equating to z gives same plane as in (a)

4. c) Let $F(x,y,z) = x^2 + y^2 + z^2$ and

$$g(x,y,z) = x y^2 z^3$$

(x,y,z) critical point of g restricted to $F(x,y,z) = 1$ is equivalent to the existence of $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \lambda \nabla F(x,y,z) = \nabla g(x,y,z) \\ F(x,y,z) = 1 \end{cases} \iff \begin{cases} 2\lambda x = y^2 z^3 \\ 2\lambda y = 2xyz^3 \\ 2\lambda z = 3xy^2z^2 \\ x^2+y^2+z^2 = 1 \end{cases} \qquad (*)$$

For $x=0$, $y=0$ or $z=0$ we have $g(x,y,z) = 0$

On the other hand $g\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) = \frac{1}{3^3} = \frac{1}{27} > 0$

and $g\left(-\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) = -\frac{1}{3^3} = -\frac{1}{27} < 0$

So the maxima and minima cannot have $x=0$, $y=0$ or $z=0$ and we exclude such points from the solution of $(*)$. Then $(*)$ gives from the first three equations:

$$\frac{y^2z^3}{x} = 2xz^3 = 3xy^2z$$

$\Rightarrow y^2 = 2x^2$ and $z^2 = \frac{3}{2}y^2 = 3x^2$. Filling this into $F(x,y,z) = 1$ gives

$x^2 + 2x^2 + 3x^2 = 1$. So $x^2 = \frac16$, $y^2 = \frac13$, $z^2 = \frac12$.

For $(x,y,z) = \left(\frac{1}{\sqrt6}, \pm\frac{1}{\sqrt3}, \pm\frac{1}{\sqrt2}\right)$, $g(x,y,z) = \frac{1}{6\sqrt3}$

$\rightsquigarrow$ maxima

For $(x,y,z) = \left(-\frac{1}{\sqrt6}, \pm\frac{1}{\sqrt3}, \pm\frac{1}{\sqrt2}\right)$, $g(x,y,z) = -\frac{1}{6\sqrt3}$

$\rightsquigarrow$ minima

5. a) $\nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax\sin(\pi y) & x^2\cos(\pi y) + bye^{-z} & y^2 e^{-z} \end{vmatrix}$

$$= \left(2ye^{-z} + bye^{-z}\right) i + (0-0) j + \left(2x\cos(\pi y) - ax\pi \cos(\pi y)\right) k$$

$$\overset{!}{=} 0$$

$$\Rightarrow b = -2, \quad a = \frac{2}{\pi}$$

b) let f be a potential function

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{2}{\pi} x \sin(\pi y) \qquad (*)$$

$$\frac{\partial f}{\partial y} = x^2 \cos(\pi y) - 2ye^{-z} \qquad (**)$$

$$\frac{\partial f}{\partial z} = y^2 e^{-z} \qquad (***)$$

Integrating $(*)$ w.r.t. x gives

$$f(x,y,z) = \frac{1}{\pi} x^2 \sin(\pi y) + g(y,z)$$

Pulling in $(**)$ gives:

$$x^2 \cos(\pi y) + \frac{\partial g}{\partial y} = x^2 \cos(\pi y) - 2ye^{-z}$$

$$\Rightarrow \frac{\partial g}{\partial y} = -2ye^{-z}.$$ Integrating w.r.t. y

gives $g(y,z) = -y^2 e^{-z} + h(z).$

$$\Rightarrow f(x,y,z) = \frac{1}{\pi} x^2 \sin(\pi y) - y^2 e^{-z} + h(z).$$

Inserting in (***) gives

$$y^2 e^{-z} + h'(z) = y^2 e^{-z}$$

$$\Rightarrow h(z) = c \;,\; c \in \mathbb{R} \text{ const.}$$

$$\Rightarrow f(x,y,z) = \frac{1}{\pi} x^2 \sin(\pi y) - y^2 e^{-z} + c$$

c) $r(0) = \vec{i}$

$r(\frac{\pi}{2}) = \vec{j}$

by Fundamental Th^m of Line Integrals

$$\int_{C_1} \vec{F} \cdot dr = f(r(\pi)) - f(r(0)) = f(\vec{j}) - f(\vec{i})$$

$$= -1 + c - c = -1$$

Solutions

1. Solutions to Ex. 1.
2. Solutions to Ex. 2.
3. Solutions to Ex. 3.
4. Solutions to Ex. 4.
5. Solutions to Ex. 5.
6. (a) Note that the boundary of D is given by the union of the spheres S_5 and S_7. The region D induces an orientation on S_7 that corresponds with the orientation of an outward-pointing normal, and it induces an orientation on S_5 that corresponds with the orientation of an *inward*-pointing normal (the orientation of D is outward, which means normal vectors pointing to the origin on S_5). Thus Gauß's theorem gives

$$\iiint_D \nabla \cdot \mathbf{F} \, \mathrm{d}V = \oiint_{S_7} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \oiint_{S_5} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 7a + b - (5a + b) = 2a.$$

   Here the minus sign compensates for the fact that the formula only holds for a sphere with outward orientation.

   (b) For $r > 0$, let $D_r = \{(x, y, z) | \, x^2 + y^2 + z^2 \leq r^2\}$. Then the boundary of D_r is S_r, and D_r provides S_r with the orientation from an outward-pointing normal. Hence Gauß's theorem gives

$$ar + b = \oiint_{S_r} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{D_r} \nabla \cdot \mathbf{F} \, \mathrm{d}V = \iiint_{D_r} \nabla \cdot (\nabla \times \mathbf{G}) \, \mathrm{d}V = 0,$$

   since $\nabla \cdot (\nabla \times \mathbf{G}) = 0$. It holds in particular for $r = 1$, so that $b = -a$. Then $ar - a = a(r - 1) = 0$ for all r, which implies that $a = 0$, so that also $b = 0$. Thus we see that if $\mathbf{F}$ is the curl of another vector field, we must have $a = b = 0$.

   Another way to see that $ar + b = 0$ for all $r > 0$ is to use Stokes. Note that S_r has empty boundary, so that Stokes gives

$$ar + b = \oiint_{S_r} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \oiint_{S_r} \nabla \times \mathbf{G} \cdot \mathrm{d}\mathbf{S} = \oint_{\emptyset} \mathbf{G} \cdot \mathrm{d}\mathbf{r} = 0.$$